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Approximate Methods for the Computation of Wave Propagation in Nonuniform Media

by

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Summary

Approximate methods for the computation of propagation of plane waves through a nonuniform dielectric slab, scattering by a nonuniform dielectric cylinder, and a conducting cylinder surrounded by a nonuniform dielectric are considered. The differential equations involved can be solved by Taylor's method, by the method of collocation, and by the method of least squares for this purpose. The dielectric slab of exponentially varying permittivity is being treated by both the rigorous and the approximate methods. The results of the approximate method are being compared with the rigorous solutions for obtaining an indication of the accuracy of the approximations.

Author

Introduction

The scattering of plane waves traveling in nonuniform media is of current interest. The problem can be dealt with by solving the wave equation derived from Maxwell's equations. The WKB method[1], Born Approximation,[2,3] and partial wave method[4] have been applied to solve this problem. However, each one of these methods has its limitations of practicality for specific values of the parameters. Either the permittivity has a small gradient or the far field is being considered only.

In this report, attempts have been made to obtain a solution for the propagation of plane waves through a nonuniform dielectric slab and waves scattered by a dielectric cylinder and a conducting cylinder surrounded by a dielectric without any limitation. The Taylor method and the method of collocation[5] are adopted to solve the problem. The first method is simpler but only good for low frequency propagation through an infinite dielectric slab. The second method is applicable to both cases, however, it is more complicated. The normal scattering by an infinite slab of exponentially varying permittivity is considered as an example to demonstrate the accuracy of Taylor's and the collocation method.

Formulation

Throughout this report, only time harmonic varying fields are being considered with the time dependence exp(jwt). All media are assumed to be linear, isotropic and lossless with a uniform permeability $\mu = \mu_o$. The permittivity ϵ is a regular function only of one dependent variable. The variation of the permittivity is confined to a finite region in the varying coordinate. The scattering of a plane wave by an infinite dielectric slab and an infinite dielectric cylinder will be considered separately.

I. Propagation of a Plane Wave Through an Infinite Dielectric Slab

Consider the oblique incidence of a plane wave on a plane dielectric slab as shown in Figure 1. The perpendicular polarization means that the electric field intensity vector is perpendicular to the plane of incidence [see Fig. 1(a)].

If the electric field intensity vector is parallel to the plane of incidence as indicated in Figure 1 (b), the wave is said to have parallel polarization. Suppose that the normalized incident fields for the perpendicular polarization and the parallel polarization are given by

$$E^{i}_{1y} = \exp.\ (jk_x x + jk_z z)\ ,$$

$$H^{i}_{2y} = \exp.\ (jk_x x + jk_z z)$$

respectively. The subscript 1 is used to indicate that the quantities are related to the perpendicular polarization; the parallel polarization is indicated by the subscript 2. The propagation constants in the x- and z- directions are

$$K_x = K_o \cos \alpha\ ,$$

$$K_z = K_o \sin \alpha\ ,$$

where $K^2_o = \omega^2 \mu_o \epsilon_o$ and the angle α is the angle of incidence. The total fields in Regions I and III (see Fig. 1) can be written as

$$E^{I}_{1y} = \exp.\ (jk_x x + jk_z z) + R_1 \exp.\ (-jk_x x + jk_z z),$$

$$E^{III}_{1y} = T_1 \exp.\ (jk_x x + jk_z z)\ ,$$

$$H^{I}_{2y} = [\exp.\ (jk_x x + jk_z z) - R_2 \exp.\ (-jk_x x + jk_z z)] / Z_o\ ,$$

$$H^{III}_{2y} = (T_2 / Z_o)\ \exp.\ (jk_x x + jk_z z)\ ,$$

where the wave impedence is $Z_o = (\mu_o / \epsilon_o)^{1/2}$, the parameters T and R are the transmission and the reflection coefficients respectively. From Maxwell's equations, the electric field intensity inside the dielectric slab of the perpendicular polarization may be represented by

$$E^{\parallel}_{1y} = [\, A_1 F_1 (x) + B_1 G_1 (x) \,] \exp (jk_z z) ,$$

where the constants A and B are determined by the boundary conditions, and the functions F_1 and G_1 are two linearly independent particular solutions of the following differential equation:

$$\psi'' (x) + k_o^2 [\, \epsilon_r (x) - \sin^2 \alpha \,] \, \psi (x) = 0 . \tag{1}$$

The derivatives of the function with respect to the argument are denoted by "primes". The relative permittivity, ϵ_r is a function of x only. The corresponding magnetic field intensity of parallel polarization is given by

$$H^{\parallel}_{2y} = [\, A_2 F_2 (x) + B_2 G_2 (x) \,] \exp (jk_z z) ,$$

where the functions F_2 and G_2 are linearly independent and satisfy the differential equation:

$$\phi'' (x) - [\, \epsilon_r' (x)/ \epsilon_r(x) \,] \, \phi' (x) + k_o^2 [\, \epsilon_r (x) - \sin^2\alpha \,] \, \phi (x) = 0. \tag{2}$$

If the functions F_1, G_1, F_2 and G_2 can be obtained by any mean from Eqs. (1) and (2) respectively, all other field components of both cases and in all three regions can be derived from Maxwell's equations accordingly. The main interests of the present problem are the transmission coefficient T and the reflection coefficient R. The conventional method can be used to evaluate these two quantities in a straightforward manner as follows. Equating the tangential field components at two slab surfaces, i.e. at x = o, and a, yields four linear inhomogeneous algebraic equations of four unknowns T, R, A, and B for both cases. By solving these four equations, the transmission and reflection coefficients for both polarizations are given by

$$T_1 = (2j/k_x D_1)\,[F_1'(o)\,G_1(o) - F_1(o)G_1'(o)]\ \exp(jk_x a), \tag{3}$$

$$R_1 = D_1^{-1}\ \{F_1(o)\,G_1(a) - F_1(a)\,G_1(o) - k_x^{-2}\,[F_1'(o)\,G_1'(a) - F_1'(a)\,G_1'(o)]$$

$$+ jk_x^{-1}\ [F_1'(o)\,G_1(a) + F_1(o)\,G_1'(a) - F_1'(a)\,G_1(o) - F_1(a)\,G_1'(o)]\}\ \exp.(2jk_x a), \tag{4}$$

$$T_2 = [2j/k_x\,\epsilon_r(o)\,D_2]\ [F_2'(o)\,G_2(o) - F_2(o)\,G_2'(o)]\ \exp.\ (jk_x a), \tag{5}$$

$$R_2 = D_2^{-1}\{F_2(a)\,G_2(o) - F_2(o)\,G_2(a) + [F_2'(o)\,G_2'(a) - F_2'(a)\,G_2'(o)]\,[k_x^2\,\epsilon_r(o)\,\epsilon_r(a)]^{-1}$$

$$+ jk_x^{-1}\ [\ (F_2(a)\,G_2'(o) - F_2'(o)\,G_2(a)]\ /\ \epsilon_r(o)$$

$$+ (F_2'(a)\,G_2(o) - F_2(o)\,G_2'(a))/\epsilon_r(a)]\}\ \exp(2jk_x a)\ . \tag{6}$$

where

$$D_1 = F_1(o)\ G_1(a) - F_1(a)\,G_1(o) + k_x^{-2}\,[F_1'(o)\,G_1'(a) - F_1'(a)\,G_1'(o)]$$

$$+ jk_x^{-1}\,[F_1'(o)\,G_1(a) - F_1(o)\,G_1'(a) + F_1'(a)\,G_1(o) - F_1(a)\,G_1'(o)]\ ,$$

$$D_2 = F_2(o)\,G_2(a) - F_2(a)\,G_2(o) + [k_x^2\ \epsilon_r(o)\,\epsilon_r(a)]^{-1}[F_2'(o)\,G_2'(a) - F_2'(a)\,G_2'(o)]$$

$$+ jk_x^{-1}\ \{\ [\ F_2'(o)\,G_2(a) - F_2(a)\,G_2'(o)\,]/\epsilon_r(o)$$

$$+\ [\ F_2'(a)\,G_2(o) - F_2(o)\,G_2'(a)\,]/\epsilon_r(a)\}\ .$$

In the event of an infinite perfectly conducting plane located at $x = 0$, no wave can be transmitted into region III, i.e., $T_1 = T_2 = 0$. Furthermore,

$$A_1 F_1(o) + B_1 G_1(o) = 0 ,$$

$$A_2 F'_2(o) + B_2 G'_2(o) = 0.$$

Under these boundary conditions, the reflection coefficients are obtained as follows:

$$R_1 = [(-S'_1 + jS''_1)/(S'_1 + jS''_1)] \exp. (2jk_x a), \tag{7}$$

$$R_2 = [(S'_2 - jS''_2)/ (S'_2 + jS''_2)] \exp (2jk_x a), \tag{8}$$

where

$$S'_1 = F'_1(a) G_1(o) - F_1(o) G'_1(a) ,$$

$$S''_1 = k_x [F_1(a) G_1(o) - G_1(a) F_1(o) ,$$

$$S'_2 = F'_2(a) G_2(o) - F_2(o) G'_2(a) ,$$

$$S''_2 = k_x \epsilon_r(a) [F_2(a) G_2(o) - F_2(o) G_2(a)] .$$

With the knowledge of the transmission and the reflection coefficients, the wave propagation in Regions I and III are completely specified. The remaining problem is how to obtain the general solutions of the two linear homogeneous differential equations [Eqs. (1) and (2)] within the region $0 \leq x \leq a$.

II. Scattering of a Plane Wave by a Nonuniform Dielectric Cylinder

The geometry under consideration is a dielectric cylinder of radius a whose axis is colinear with the z-axis of the cylindrical coordinate system as shown in Fig. 2. The relative permittivity of the dielectric is a function of radius only, i.e., it may be expressed by $\epsilon_r(\rho)$, where ρ is the radial coordinate. Again, two cases will be considered, namely, the perpendicular polarization and the parallel polarization. The z-component of the incident fields in both cases are given by

$$H^i_{1z} = (\cos \alpha/Z_o) \exp.(jk_x x + jk_z z) ,$$

$$E^i_{2z} = \cos \alpha \exp. (jk_x x + jk_z z) ,$$

where the propagation constants, k_x and k_z, and the wave impedance, Z_o, are given as before. The angle, α , is the angle of incidence. Using the well known wave transformation, the factor, $\exp(jk_x x)$ can be expressed in terms of Bessel functions of the first kind and the cosine functions of the angular coordinate, θ. That is

$$\exp. (jk_x x) = J_o(\xi) + 2 \sum_{n=1}^{\infty} j^n J_n(\xi) \cos \theta ,$$

where $x = \rho \cos\theta$, the argument is

$$\xi = \rho k_x \cos \alpha .$$

Since the cylinder is assumed to have infinite length and the permittivity is uniform in the z-direction, the resultant fields must be periodic in the z-direction and varying according to the factor $\exp(jk_z z)$. The total fields (incident plus scattering) in air may be written as

$$H_{1z} = (\cos \alpha/Z_o) \{ J_o(\xi) + a_o H_o^{(2)}(\xi)$$

$$+ 2 \sum_{n=1} [j^n J_n(\xi) + a_n H_n^{(2)}(\xi)] \cos n\theta \} \exp.(jk_z z),$$

$$E_{2z} = (\cos\alpha) \exp. (jk_z z) \{ J_o(\xi) + b_o H_o^{(2)}(\xi)$$

$$+ 2 \sum_{n=1} [j^n J_n(\xi) + b_n H_n^{(2)}(\xi)] \cos n\theta \} ,$$

where the function, $H_n^{(2)}$ is the n^{th} order Hankel function of the second kind. The constants, a_n and b_n are detemined by the boundary conditions. The corresponding fields inside the dielectric cylinder are given by

$$H_{1z}^d = \exp.(jk_z z) \sum_{n=o} c_n \bar{\psi}_n(\rho) \cos n\theta ,$$

$$E_{2z}^d = \exp.(jk_z z) \sum_{n=o} d_n \bar{\phi}(\rho) \cos n\theta ,$$

where the functions $\bar{\psi}_n$ and $\bar{\phi}_n$ are the regular particular solutions of the following differential equations respectively:

$$\bar{\psi}_n''(\rho) + [\frac{1}{\rho} - \epsilon_r'(\rho)/W(\rho)] \bar{\psi}_n'(\rho) + [k_o^2 W(\rho) - \frac{n^2}{\rho^2}] \bar{\psi}_n(\rho) = 0. \quad (9)$$

$$\bar{\phi}_n''(\rho) + \{\frac{1}{\rho} - \epsilon_r'(\rho)[\frac{1}{\epsilon_r(\rho)} - \frac{1}{W(\rho)}]\} \bar{\phi}_n(\rho) + [k_o^2 W(\rho) - \frac{n^2}{\rho^2}] \bar{\phi}_n(\rho) = 0 , \quad (10)$$

where $W(\rho) = \epsilon_r(\rho) - \sin^2\alpha$. With the knowledge of the z-component of the magnetic or the electric intensity, the θ -component of the corresponding electric or magnetic field can be obtained by

$$E_1 = [j\omega\mu_o/k_o^2 W(\rho)] \frac{\partial H_{1z}}{\partial \rho}$$

$$H_2 = [-j\omega\epsilon/k_o^2 W(\rho)] \frac{\partial E_{1z}}{\partial \rho}$$

respectively. The scattering amplitudes, a_n and b_n are evaluated by equating the tangential components at the surface of the cylinder and putting the tems of the same angular dependent equal to zero. The two algebraic equations for the scattering amplitudes a_n and c_n for perpendicular polarization, b_n and d_n for parallel polarization, can then be solved for each n. They are

$$a_n = \frac{\cos^2\alpha\,\bar{\psi}'_n(a)\,J_n(\xi_o) - k_x W(a)\,\bar{\psi}_n(a)\,J'_n(\xi_o)}{k_x W(a)\,\bar{\psi}_n(a)\,H'_n(\xi_o) - \cos^2\alpha\,\bar{\psi}'_n(a)\,H_n(\xi_o)}\,, \tag{11}$$

$$b_n = \frac{\epsilon_r(a)\cos^2\alpha\,\bar{\phi}'_n(a)\,J_n(\xi_o) - k_x W(a)\,J'_n(\xi_o)\,\bar{\phi}_n(a)}{k_x W(a)\,\bar{\phi}_n(a)\,H'_n(\xi_o) - \epsilon_r(a)\cos^2\alpha\,\bar{\phi}'_n(a)\,H_n(\xi_o)}\,, \tag{12}$$

where $\xi_o = ak_o \cos\alpha$. The solution will be complete if the functions, $\bar{\psi}_n$ and $\bar{\phi}_n$ are known. For the case of a conducting cylinder of radius b coated by a nonuniform dielectric up to radius a, Eqs. (11) and (12) are applicable except that the function $\bar{\psi}_n$ and $\bar{\phi}_n$ are the general solutions of Eqs. (9) and (10) respectively. An additional boundary condition for the perpendicular polarization is $\bar{\psi}'_n(b) = 0$; the corresponding boundary condition for the parallel polarization is $\bar{\phi}_n(b) = 0$.

It should be noted that for arbitrarily polarized waves, the incident field can be resolved into two waves, one polarized in the perpendicular direction, and the other with parallel polarization. The resultant fields are given by the sum of these two solutions.

Solutions of the Differential Equations

As shown above, the scattering of plane waves by a plane or a cylindrical object can be calculated if the particular solutions of the differential Equations (1), (2), (9), and (10) can be found. Since the nature of the solutions of the plane case are quite different from those of the cylindrical case, the two cases will be treated separately. We begin with the plane case.

I. Solutions for the Differential Equation of a Slab

It is obvious that Eqs. (1) and (2) are of the same type and can be represented by

$$U''(x) + p(x)\,U'(x) + q(x)\,U(x) = 0 \tag{13}$$

where p (x) and q (x) are regular functions within the region under consideration, i.e., p (x) and q (x) have no singular point within the region $0 \le x \le a$. By Taylor's method, the general solution of Eq. (13) may be written as

$$U(x) = U(0) + U'(o)\,x + \frac{U''(o)}{2}\,x^2 + \frac{U'''(o)}{3}\,x^3 + \ldots . \tag{14}$$

if the series is convergent in $0 \le x \le a$. To assure that Eq. (14) is the general solution of Eq. (13), it is suggested that

$$U(o) = A, \qquad U'(o) = B. \tag{15}$$

Since Eq. (14) satisfies Eq. (13), the second derivative of the function U' evaluated at x = 0 can be obtained by substituting Eq. (15) into (13). That is

$$U''(o) = -q(o)\,A - p(o)\,B$$

The higher order terms are obtained by taking the derivative with respect to x of Eq. (13) a certain number of times. For example, taking the derivative once yields

$$U'''(o) = -[p(o)\,q(o) + q'(o)]\,A + [p^2(o) + q(o) - p'(o)]\,B .$$

In this way, the higher order terms $U^{lv}(0)$, $U^{v}(0)$, can be calculated accordingly. Substituting all of these into Eq. (14) yields the general solution for Eq. (13) of the following form

$$U(x) = A\,\Sigma\,A_n x^n + B\,\Sigma\,B_n x^n$$

with known values of A_n's and B_n's. The convergence of the series depends on the width of the slab and the operating frequency. Usually, it is useful only at low frequency and for a narrow slab. For higher frequencies and wide slab, the method of collocation which is discussed in the following consideration seems to be more practical.

Suppose that U_1 and U_2 are two linearly independent particular solutions of Eq. (13). It is well known that any function can be expressed by an even and an odd functions. Thus, let

$$U_1 = U_{1e} + U_{1o}$$

$$U_2 = U_{2e} + U_{2o}$$

where the subscripts e and o indicate that the function is even or odd. It is easy to show that the two functions $U_{1e} + U_{2e}$ and $U_{1o} + U_{2o}$ are two linearly independent solutions of Eq. (13). Additionally, any regular function within a finite region can be expressed with validity in this region by a Fourier series. Hence, the two linearly independent particular solutions of Eq. (13) can be written as

$$U_e(x) = U_{1e} + U_{2e} = \Sigma A_n \cos L_{1n} x \ ,$$

$$U_o(x) = U_{1o} + U_{2o} = \Sigma B_n \sin L_{2n} x \ ,$$

where $L_{1n} = n\pi/v_1 a$, $L_{2n} = n\pi/v_2 a$. Two functions $U_e(x)$ and $U_o(x)$ are valid within $0 \leq x \leq a$. The dimensionless quantity v_1 and v_2, to be determined by the differential equation, are two real numbers greater than or equal to unity. They are equal if the summations of these two series are summing the index n from zero to infinity. However, if, as an approximation, the series have finite numbers of terms, v_1 and v_2 may have different values. Let

$$U_e(x) = \sum_{n=o}^{N} A_n \cos L_{1n} x, \tag{16a}$$

$$U_o(x) = \sum_{m=o}^{M} B_m \sin L_{2m} x, \tag{16b}$$

where n, m are integers. Substituting Eqs. (16) into Eq. (13) separately yields

$$\sum_{n=o}^{N} \{[q(x) - L_{1n}^2] \cos L_{1n} x - p(x) L_{1n} \sin L_{1n} x\} A_n = 0, \tag{17a}$$

$$\sum_{m=1}^{M} \{[q(x) - L_{2m}^2] \sin L_{2m} x + p(x) L_{2m} \cos L_{2m} x\} B_m = 0. \tag{17b}$$

Obviously, if Eqs. (16) are two particular solutions of (13), Eqs. (17) have to be satisfied at all points of the region $0 \le x \le a$ (there is no objection if Eqs. (17) are also satisfied at $x > a$). But, for the purpose of approximation, the method of collocation[5] requires the equality to be fulfilled only at N + 1 points for Eq. (17a) and M points for Eq. (17b). In the following only Eq. (17b) will be considered. Suppose that Eq. (17b) is satisfied at M points, namely, $0 \le x_1 \le x_2 \le x_3 \ldots \le x_m \le a$. There are many choices of the points. Usually, it is convenient to choose equal space between the points. For each point, Eq. (17b) is an algebraic equation of M unknowns, B_m. That is

$$\sum_{m=1}^{M} \{[q(x_i) - L_{2m}^2] \sin L_{2m} x_i + p(x_i) L_{2m} x_i\} B_m = 0, \tag{18}$$

where i = 1, 2, 3, ..., M. In order to have nontrivial solutions for the B_m's, the determinant formed from the coefficients which depends on the value of v_2 must be zero. By this condition, the value of v_2 is then determined. With the knowledge of v_2, the expansion coefficients B_m can be calculated from the system of M-1 equations in terms of a B_r, which is largest among the B_m's. There are many roots from the determinant of v_2. Taking the convergence into account, the suitable value is the smallest root which is greater than or equal to unity.

Similar procedures lead to solutions of v_1 and the corresponding expansion coefficients, A_n of Eq. (17a). Note that if $p(x) = 0$, $v_2 = 1$ is always a root of the determinant. But it is not necessarily a suitable value.

It should be mentioned that the method of least squares[5] is applicable to this case too. Multiplying Eq. (17a) by $\cos L_{1r}x$, and Eq. (17b) by $\sin L_{2s}x$ and integrating from 0 to a with respect to x yields

$$\sum_{n=o}^{N} A_n \int_o^a \cos L_{1r}x \; \{ [q(x) - L^2_{1n}] \cos L_{1n}x - p(x) L_{1n} \sin L_{1n}x \} \, dx = 0, \quad (19a)$$

$$\sum_{m=1}^{M} B_m \int_o^a \sin L_{2s}x \; \{ [q(x) - L^2_{2m}] \sin L_{2m}x + p(x) L_{2m} \cos L_{2m}x \} \, dx = 0. \quad (19b)$$

The integrals in Eqs. (19) can be approximated by a weighted sum of the relevant ordinates at k points. That is

$$\sum_{n=o}^{N} A_n \sum_{i=1}^{k} D_i \cos L_{1r}x_i \{ [q(x_i) - L^2_{1n}] \cos L_{1n}x_i - p(x_i) L_{1n} \operatorname{Sin} L_{1n}x_i \} = 0, \quad (20a)$$

$$\sum_{m=o}^{M} B_m \sum_{i=1}^{k} D_i \sin L_{2s}x_i \{ [q(x_i) - L^2_{2m}] \sin L_{2m}x_i + p(x_i) L_{2m} \cos L_{2m}x_i \} = 0, \quad (20b)$$

where $r = 0, 1, 2, 3, \ldots N$, $s = 1, 2, 3, \ldots M$; and $x_i = ia/k$ According to the Trapezoidal rule,[6] the weighing coefficients, D_i are given by

$$(D_o, D_1, D_2, D_3, \ldots D_{k-1}, D_k) = \frac{a}{k} \left(\frac{1}{2}, 1, 1, 1, \ldots, 1, \frac{1}{2} \right).$$

According to Simpson's one-third rule,[6] they may be written as

$$(D_o, D_1, D_2, D_3, \ldots D_{k-1}, D_k) = \frac{a}{3k} \; (1, 4, 2, 4, 2, \ldots, 4, 1).$$

Other rules can be used as well. Eqs. (20) again, are systems of linear, homogeneous algebraic equations which can be exploded to find the suitable values for v_1, v_2,

A_n's and B_m's. The solutions of the method of collocation will be different with a different choice of the points where Eqs. (17) are satisfied. This phenomena does not exist in the method of least squares which is considerably more accurate, but also more complicated.

By means of a high speed electronic computer, it is easy to evaluate a determinant or to solve a system of linear algebraic equations. The methods of collocation and of least squares are hence practical to calculate the transmission and reflection coefficients.

II. Solutions for the Cylindrical Problem

Observe that the differential equations of both perpendicular and parallel polarizations are of the same form. That is

$$V''_n(\rho) + \left[\frac{1}{\rho} + p(\rho) \right] V'_n(\rho) + \left[q(\rho) - \frac{n^2}{\rho^2} \right] V_n(\rho) = 0 \qquad (21)$$

where $p(\rho)$ and $q(\rho)$, again, are regular functions of ρ within the region $0 \leq \rho \leq a$. If $p(\rho) \neq 0$, Taylor's method is not applicable to Eq. (21). Hence, only the methods of collocation and of the least squares are practical. They are considered in the following section.

In order to fulfill the requirement that all fields are finite at $\rho = 0$, the function $V_n(\rho)$ of Eq. (21) can be expressed as

$$V_n(\rho) = \sum_{m=1} C_{nm} J_n(\alpha_{nm} \zeta), \qquad (22)$$

where $\zeta = \rho / \upsilon a$, $J_n(\alpha_{nm}) = 0$, and J_n is the n^{th} order Bessel function of the first kind. The dimensionless parameter, υ, to be determined by the differential equation, is a real number which is greater than or equal to unity. To consider an approximate solution, the summation of m in Eq. (22) may be taken from 1 to M such that Eq. (22) can be a good approximation. Substitute Eq. (22) into (21) and notice that the relationship

$$J''_n(x) + \frac{1}{x} J'_n(x) + \left(1 - \frac{n^2}{x^2}\right) J_n(x) = 0,$$

yields

$$\sum_{m=o}^{M} C_{nm} \{ \frac{\alpha_{nm}}{ua} p(\rho) J'_n (\alpha_{nm} \zeta) + [q(\rho) - (\frac{\alpha_{nm}}{ua})^2 J_n (\alpha_{nm} \zeta)] \} = 0 . \quad (23)$$

Note that Eq. (23) is similar to Eq. (17b). The differences are that the trigonometric functions are replaced by the Bessel functions and $m\pi$ is replaced by α_{nm}. Evidently, the method of collocation and the method of least squares are applicable to the present problem. The same procedures can be used to determine the suitable value of u and the expansion coefficients, C_{nm}. In other words, if all the sin are replaced by J_n, cos by J'_n, $m\pi$ by α_{nm}, and B_m by C_{nm} in Eqs. (16b), (17b), (18), (19b) and (20b), then all these equations and the associated statements are valid for the solution of Eq. (21).

For the case of a conducting cylinder of radius b coated by a nonuniform dielectric up to radius a, all $C_{nm} J_n$ in Eqs. (22) - (23) are replaced by $C_{nm} J_n + D_{nm} N_n$, and $C_{nm} J'_n$ by $C_{nm} J'_n + D_{nm} N'_n$, where the function N_n is the n^{th} order Bessel function of the second kind. From the boundary conditions, the constants D_{nm} for perpendicular polarization is given by

$$D_{nm} = - C_{nm} J'_n (\alpha_{nm} \zeta_b) / N'_n (\alpha_{nm} \zeta_b) ,$$

where $\zeta_b = b/ua$. For parallel polarization, it is

$$D_{nm} = - C_{nm} J_n (\alpha_{nm} \zeta_b) / N'_n (\alpha_{nm} \zeta_b) .$$

The same method for finding the parameter u and the expansion coefficients, C_{nm}, is the same as discussed previously.

Examples

The accuracy of the approximate methods discussed previously can be demonstrated by comparison with the rigorous solution. This can be done by considering a lossless slab of exponentially varying permittivity, i.e.

$$\epsilon_r(x) = \epsilon_r e^{-x/a}$$

where ϵ_r is a constant, and a is the width of the slab. The rigorous solution of the electric field intensity in the dielectric slab for the normal incident fields ($\alpha = 0$) is given by

$$E_y = AJ_o(\eta) + BN_o(\eta),$$

where the argument is

$$\eta = 2k_r a e^{-x/2a},$$

$$k_r^2 = k_o^2 \epsilon_r,$$

and the functions J_o and N_o are the zero order Bessel functions of the first and second kind respectively. The solution of Taylor's method calculated up to the six power of x is given by

$$E_y^T = A\,[\,1 - \frac{k_r^2}{2!}x^2 + \frac{k_r^2}{3!}\frac{x^3}{a} + \frac{k_r^2}{4!}(k_r^2 - a^{-2})x^4$$

$$+ \frac{k_r^2}{5!}(a^{-2} - 4k_r^2)\frac{x^5}{a} + \frac{k_r^2}{6!}x^6(11k_r^2 a^{-2} - a^{-4} - k_r^4)\,]$$

$$+ (B/a)\,[\,x - \frac{k_r^2}{3!}x^3 + \frac{2k_r^2}{4!}\frac{x^4}{a} + k_r^2\frac{x^5}{5!}(k_r^2 - 3a^{-2})$$

$$+ \frac{k_r^2}{6!}(4a^{-4} - 6k_r^2)x^6/a\,].$$

with $E_y = A$, $\frac{dE_y}{dx} = B/a$ at $x = 0$. Observe that this method is valid only when $k_r \leq 1$. Using four points ($x = 0, a/3, 2a/3$, and a for the even function; $x = a/4, a/2, 3a/4$, and a for the odd function) in the method of collocation for $k_r = \pi/a$ yields

$$E_y^c = A [\sin L_2 x + 0.082 \sin 2L_2 x + 0.00631 \sin 3L_2 x$$

$$+ 0.001919 \sin 4L_2 x]$$

$$+ B [- 0.2323 + \cos L_1 x + 0.0693 \cos 2L_1 x + 0.0212 \cos 3L_1 x$$

$$+ 0.0076 \cos 4L_1 x]$$

where $L_1 = \pi/1.307a$, $L_2 = \pi/1.376$. Because of only finite number of trigometric functions in the expansion the value of L_1 is different from that of L_2.

Table 1 lists the reflection coefficients calculated by Taylor's method and compared with the rigorous solutions at different frequencies for the case where an infinite conducting plane is located at $x = 0$. For the same geometry, the reflection coefficients calculated by the method of collocation with two and four points matching at $k_r = \pi/a$ are tabulated in Table 2. The convergence is tremendously good in this case. The error of the method of collocation by matching at two points is within 4% in comparison with the rigorous solution.

For the case without the conducting plane, the transmission and the reflection coefficients calculated by Taylor's method are tabulated in Table 3 and compared with the rigorous values. The operating frequency is chosen at $k_r = \pi/a$ and $\epsilon_r = 4$.

Table 1 - Comparison of the reflection coefficients R calculated by Taylor's Method with the rigorous solution, where

$$R = -\underline{/2\tan^{-1}\gamma}\,\exp(2jk_o a)$$

γ \ k_r	$\pi/8a$	$\pi/7a$	$\pi/6a$	$\pi/5a$
Taylor's Method	-0.405	-0.4654	-0.5508	-0.6723
Exact	-0.4096	-0.4645	-0.5466	-0.6503

Table 2 - Comparison of the reflection coefficients R calculated by the method of collocation with the rigorous solution, where

$$R = \underline{/2\tan^{-1}\gamma}\,\exp(2jk_o a)$$

M	2	4	Exact
γ	1.4483	1.4012	1.3964

M represents the number of terms in the collocation method.

Table 3 - Comparison of the transmission and reflection coefficients calculated by Taylor's Method with the rigorous values

	$T \exp(-jk_o a)$	$R \exp(-2jk_o a)$
Taylor's Method	$0.9910 \angle -19^\circ 43'$	$0.1460 \angle 112^\circ 12'$
Exact	$0.9895 \angle -19^\circ 23'$	$0.1357 \angle 112^\circ 6'$

Table - 4 Comparison of the transmission and reflection coefficients calculated by the method of collocation and the exact values

	$T \exp(-jk_o a)$	$R \exp(-2jk_o a)$
Method of collocation	$0.9161 \angle 220^\circ 5'$	$0.3802 \angle -96^\circ 40'$
Exact	$0.9209 \angle 220^\circ 23'$	$0.3795 \angle -87^\circ 51'$

Conclusion

Taylor's method, method of collocation, and method of least squares are shown applicable to the propagation of plane waves through nonuniform regions. Taylor's method is simple but limited to low frequency propagation. The method of collocation and method of least squares are practical at high frequencies. High speed electronic computers make the calculation possible at high accuracy.

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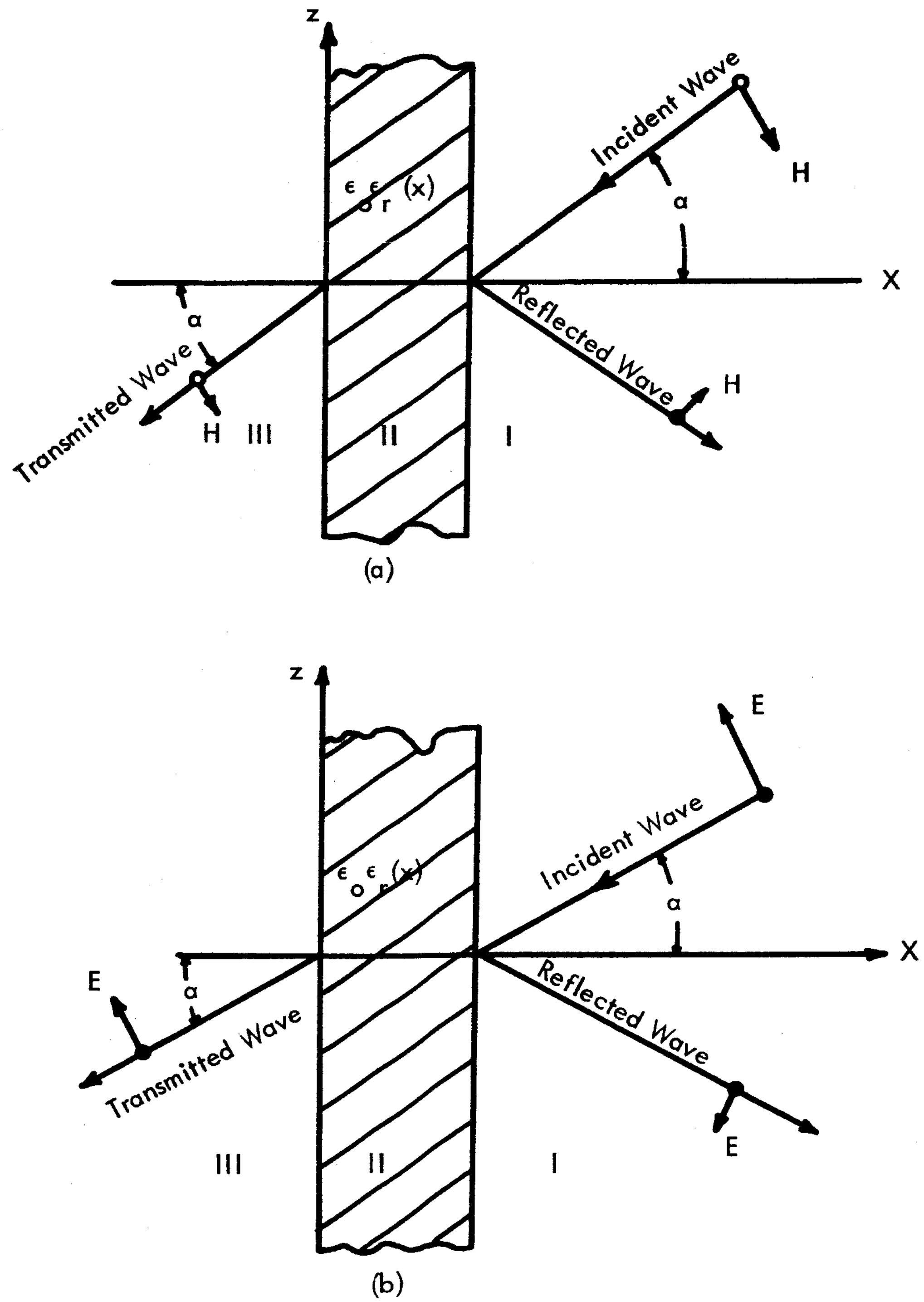

Plane-Sheath Scattering

Figure 1 - (a) Perpendicular polarization.
(b) Parallel polarization.

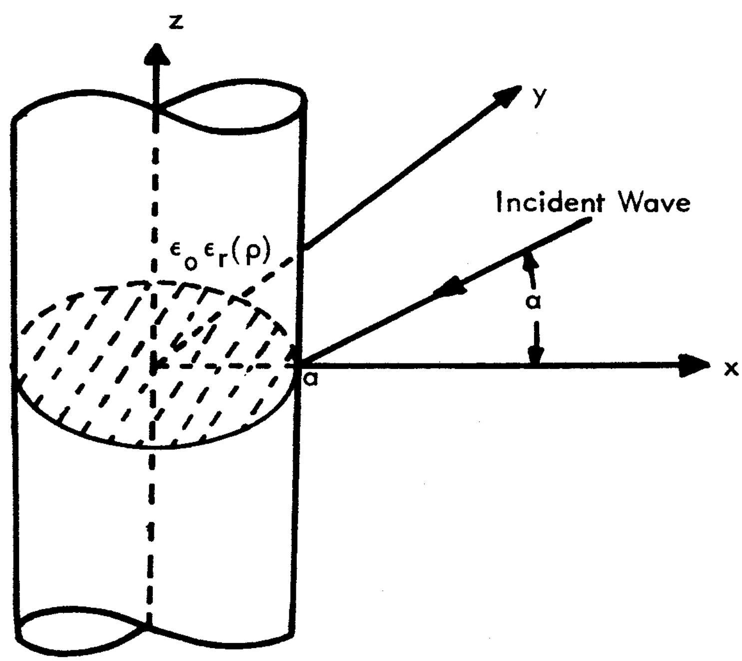

Figure 2 - Scattering of a plane wave by a cylindrical nonuniform dielectric material.